ECS315 2018/1   Part IV.2   Dr.Prapun

popular and useful

10.4   Families of Continuous Random Variables

Theorem 10.34 states that any nonnegative function $f(x)$ whose integral over the interval $(-\infty, +\infty)$ equals 1 can be regarded as a probability density function of a random variable. In real-world applications, however, special mathematical forms naturally show up. In this section, we introduce a couple families of continuous random variables that frequently appear in practical applications. The probability densities of the members of each family all have the same mathematical form but differ only in one or more parameters.

10.4.1   (Continuous) Uniform Distribution

Definition 10.44.

Compare this with a uniform RV that is discrete with support
$= \{a, b\}$
$X \sim \mathcal{U}(\{a, b\})$

an interval

- The (continuous) uniform distribution on an interval $[a, b]$, is denoted by uniform($[a, b]$) or $\mathcal{U}([a, b])$ or simply $\mathcal{U}(a, b)$.

  (a, b)
  $(a, b]$
  $[a, b)$

- Expressions that are synonymous with "X is a uniform random variable" are

  (a) "X is uniformly distributed",

  (b) "X has a uniform distribution",

  (c) and "X has a uniform density".

  To specify the support (range) of X, we may also append "on/over the interval (a, b)".

• This family is characterized by pdf of the form

$$f_X(x) = \begin{cases} 0, & x < a, x > b \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

• The constants a and b are referred to as the parameters of the uniform distribution.

10.45. Important Interpretation: A continuous uniform random variable X on the interval $[a, b]$ is just as likely to be "near" any value in $[a, b]$ as any other value.

10.46. In `MATLAB`,

(a) use `X = a+(b-a)*rand` or `X = random('Uniform',a,b)` to generate $X \sim \mathcal{U}(a, b)$,

(b) use `pdf('Uniform',x,a,b)` and `cdf('Uniform',x,a,b)` to evaluate the pdf and cdf at x, respectively.

Exercise 10.47. Show that $F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b, \\ 1, & x > b. \end{cases}$
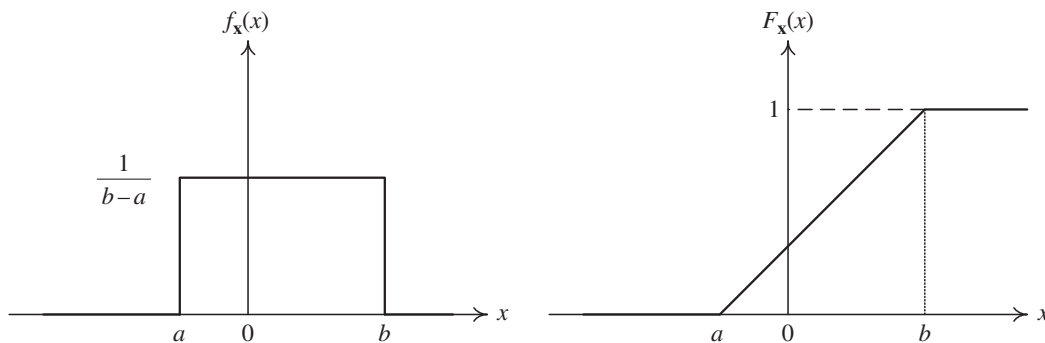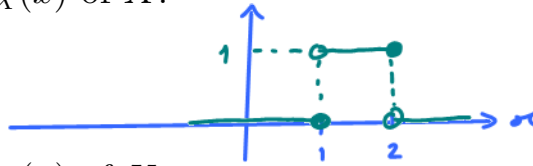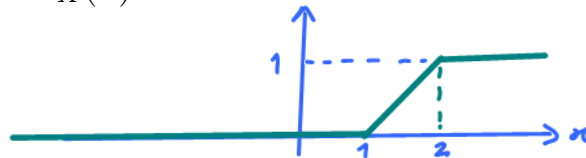

Figure 26: The pdf and cdf for the uniform random variable. [16, Fig. 3.5]

Example 10.48 (F2011). Suppose X is uniformly distributed on the interval $(1, 2)$. $(X \sim \mathcal{U}(1, 2).)$

(a) Plot the pdf $f_X(x)$ of X.

(b) Plot the cdf $F_X(x)$ of X.

10.49. The uniform distribution provides a probability model for selecting a point at random from the interval $[a, b]$.

- Use with caution to model a quantity that is known to vary randomly between a and b but about which little else is known.

Example 10.50. [9, Ex. 4.1 p. 140-141] In coherent radio communications, the phase difference between the transmitter and the receiver, denoted by Θ, is modeled as having a uniform density on $[-\pi, \pi]$.

(a) $P[\Theta \le 0] = \frac{1}{2}$
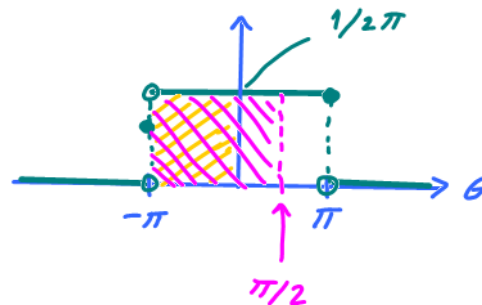
(b) $P\left[\Theta \le \frac{\pi}{2}\right] = \frac{3}{4}$

Exercise 10.51. Show that when $X \sim \mathcal{U}(a, b)$, $\mathbb{E}X = \frac{a+b}{2}$, $\text{Var } X = \frac{(b-a)^2}{12}$, and $\mathbb{E}\left[X^2\right] = \frac{1}{3}\left(b^2 + ab + a^2\right)$.

10.4.2 Gaussian Distribution

10.52. This is the most widely used model for the distribution of a random variable. When you have many independent random variables, a fundamental result called the central limit theorem (CLT) (informally) says that the sum (or the average) of them can often be approximated by normal distribution.
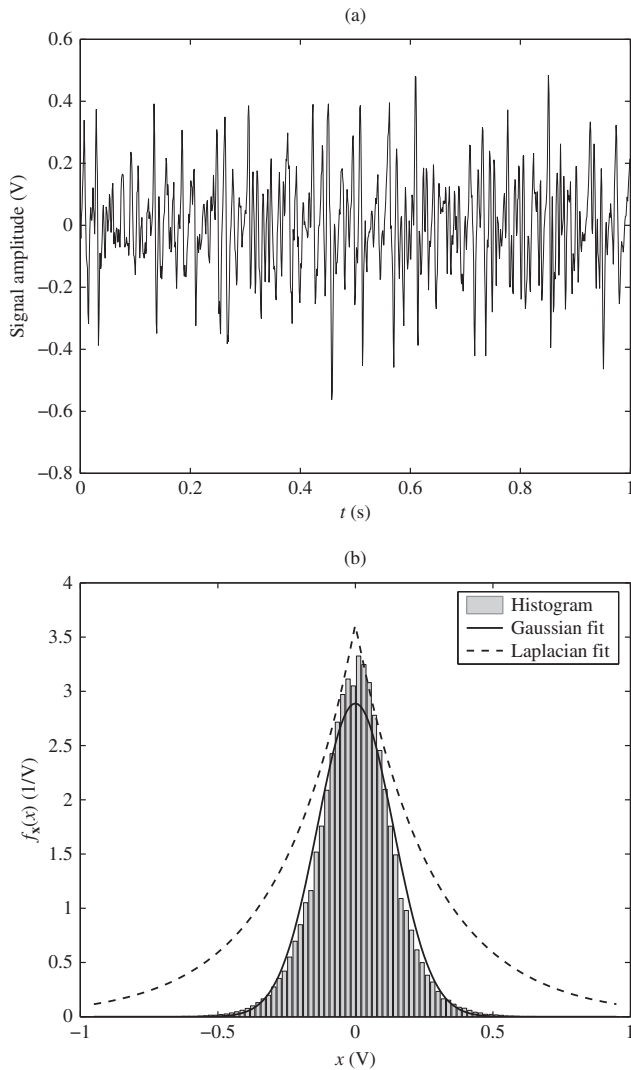

Figure 27: Electrical activity of a skeletal muscle: (a) A sample skeletal muscle (emg) signal, and (b) its histogram and pdf fits. [16, Fig. 3.14]

Definition 10.53. *Gaussian* random variables:

- Often called *normal* random variables because they occur so frequently in practice.

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- The Gaussian distribution is denoted by $\mathcal{N}(m, \sigma^2)$. It has two parameters: $m \in \mathbb{R}$ and $\sigma > 0$.

  o Caution: The second argument in $\mathcal{N}(m, \sigma^2)$ is σ^2 (not σ).

  o Several references use μ instead of m.

- This family is characterized by pdf of the form

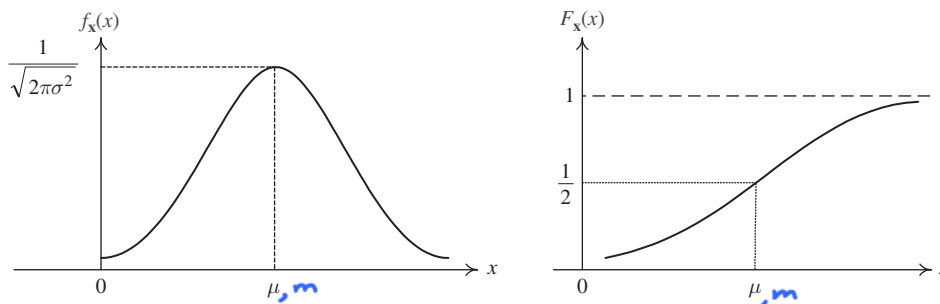$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

Figure 28: The pdf and cdf of $\mathcal{N}(\mu, \sigma^2)$. [16, Fig. 3.6]

  o In Excel, use NORMDIST(x,m,σ,FALSE).
    In MATLAB, use normpdf(x,m,σ) or pdf('Normal',x,m,σ).

  o Figure 28 and Figure 30 display the famous **bell-shaped** graphs of the Gaussian pdf. This curves are also called the *normal* curves.

- In MATLAB, use X = random('Normal',m,σ) or X = σ*randn + m to generate $X \sim \mathcal{N}(m, \sigma^2)$.

- $F_X(x)$ has no closed-form expression. However, see 10.60.

  o In MATLAB, use normcdf(x,m,σ) or cdf('Normal',x,m,σ).
  o In Excel, use NORMDIST(x,m,σ,TRUE).

10.54. $\mathbb{E}X = m$ and $\text{Var}\, X = \sigma^2$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \tfrac{1}{\sigma} f_X(x)\,dx$$
$$= \sigma^2 + m^2$$

$$\int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}\,dx \overset{155}{=} m$$

$$\text{Var}\, X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$
$$= (\sigma^2 + m^2) - m^2 = \sigma^2$$

$$\sigma_X = \sqrt{\text{Var}\, X}$$
$$= \sigma$$
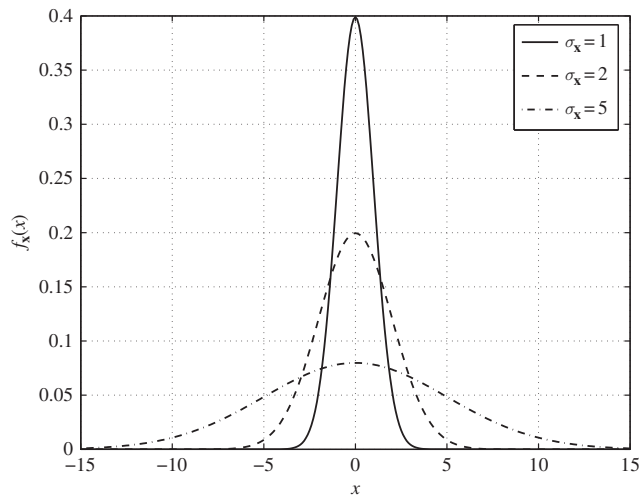
Figure 29: Plots of the zero-mean Gaussian pdf for different values of standard deviation, σ_X. [16, Fig. 3.15]

10.55. Important probabilities:

$P\left[|X - \mu| < \sigma\right] = 0.6827$;
$P\left[|X - \mu| > \sigma\right] = 0.3173$;
$P\left[|X - \mu| > 2\sigma\right] = 0.0455$;
$P\left[|X - \mu| < 2\sigma\right] = 0.9545$

These values are illustrated in Figure 30. We will see these numbers again in Example 10.61.
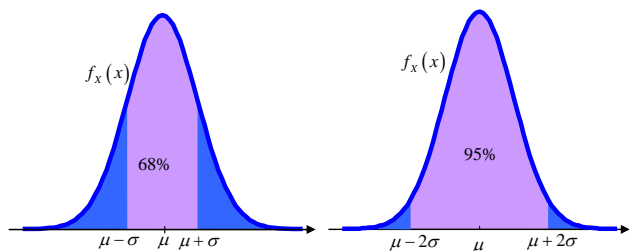
Figure 30: Probability density function of $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$. The purple areas correspond to $P\left[|X - \mu| < \sigma\right] = 0.6827$ and $P\left[|X - \mu| < 2\sigma\right] = 0.9545$, respectively.

Example 10.56. Figure 31 compares several deviation scores and the normal distribution.

(a) Standard scores have a mean of zero and a standard deviation of 1.0.

(b) Scholastic Aptitude Test scores have a mean of 500 and a standard deviation of 100.
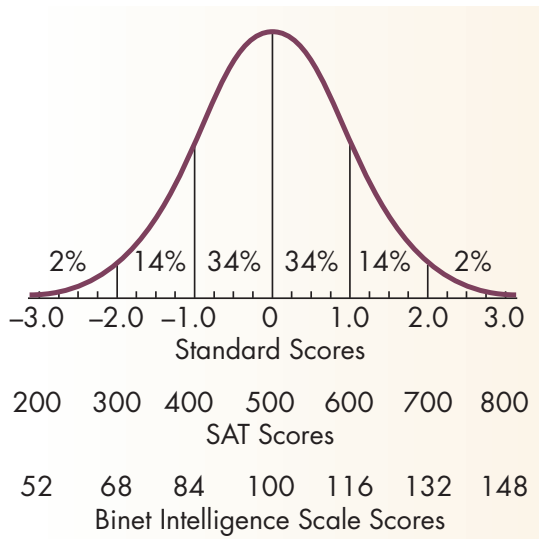
Figure 31: Comparison of Several Deviation Scores and the Normal Distribution

(c) Binet Intelligence Scale[46] scores have a mean of 100 and a standard deviation of 16.

In each case there are 34 percent of the scores between the mean and one standard deviation, 14 percent between one and two standard deviations, and 2 percent beyond two standard deviations. [Source: Beck, Applying Psychology: Critical and Creative Thinking.]

10.57. The area under a normal probability density function beyond 3σ from the mean is quite small. In fact,

$$P\left[|X - \mu| < 3\sigma\right] \approx 0.9973.$$

Therefore, approximately 99.73% of the probability of a normal distribution is within the interval $(\mu - 3\sigma, \mu + 3\sigma)$.

[46]Alfred Binet, who devised the first general aptitude test at the beginning of the 20th century, defined intelligence as the ability to make adaptations. The general purpose of the test was to determine which children in Paris could benefit from school. Binets test, like its subsequent revisions, consists of a series of progressively more difficult tasks that children of different ages can successfully complete. A child who can solve problems typically solved by children at a particular age level is said to have that mental age. For example, if a child can successfully do the same tasks that an average 8-year-old can do, he or she is said to have a mental age of 8. The intelligence quotient, or IQ, is defined by the formula:

$$\text{IQ} = 100 \times (\text{Mental Age/Chronological Age})$$

There has been a great deal of controversy in recent years over what intelligence tests measure. Many of the test items depend on either language or other specific cultural experiences for correct answers. Nevertheless, such tests can rather effectively predict school success. If school requires language and the tests measure language ability at a particular point of time in a childs life, then the test is a better-than-chance predictor of school performance.

Definition 10.58. $\mathcal{N}(0,1)$ is the **_standard_** Gaussian (normal) distribution.

- We usually use Z to denote standard Gaussian RV.
- In Excel, use NORMSINV(RAND()).
  In MATLAB, use randn.
- The standard normal cdf is denoted by $\Phi(z)$.
  - It inherits all properties of cdf.
  - Moreover, note that $\Phi(-z) = 1 - \Phi(z)$.

(handwritten annotations)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$\begin{aligned} m &= 0 \\ \sigma &= 1 \end{aligned} \quad = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$\Phi(z) \equiv F_Z(z) = P[Z \le z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\,dt$$

cdf of standard Gaussian RV
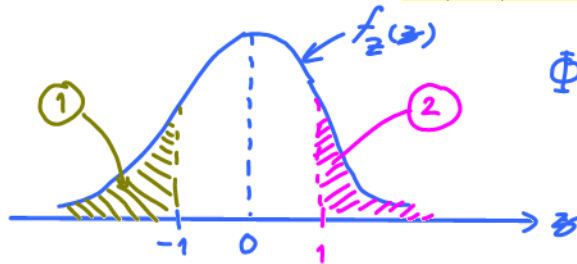
$$\Phi(-1) \equiv F_Z(-1)$$
$$\equiv P[Z \le -1] = \text{①} = \text{②}$$
$$= 1 - P[Z \le 1] \equiv 1 - F_Z(1)$$
$$= 1 - \Phi(1)$$

10.59. Relationship[47] between $\mathcal{N}(0,1)$ and $\mathcal{N}(m, \sigma^2)$.

(a) An arbitrary Gaussian random variable with mean m and variance σ^2 can be represented as $\sigma Z + m$, where $Z \sim \mathcal{N}(0,1)$.

   This relationship can be used to generate general Gaussian RV from standard Gaussian RV.

(b) If $X \sim \mathcal{N}(m, \sigma^2)$, the random variable

$$Z = \frac{X - m}{\sigma}$$

is a standard normal random variable. That is, $Z \sim \mathcal{N}(0,1)$.

- Creating a new random variable by this transformation is referred to as **_standardizing_**.
- The standardized variable is called "**_standard score_**" or "**_z-score_**".

10.60. It is impossible to express the integral of a Gaussian PDF between non-infinite limits (e.g., (20)) as a function that appears on most scientific calculators.

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[47]To be proved in 10.81.

- An old but still popular technique to find integrals of the Gaussian PDF is to refer to tables that have been obtained by numerical integration.

  - An example of such table is Table 4, which lists $\Phi(z)$ for many values of positive z.
  - For $X \sim \mathcal{N}(m, \sigma^2)$, we can show that the CDF of X can be calculated from

$$F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right).$$

$$F_X(x) \equiv P[X \le x] = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2}\,dt$$

$$\text{Let } \mu = \frac{t-m}{\sigma} \qquad d\mu = \frac{1}{\sigma}dt$$

$$= \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2}\,d\mu = F_Z\left(\frac{x-m}{\sigma}\right)$$

Example 10.61. Suppose $Z \sim \mathcal{N}(0,1)$. Evaluate the following probabilities.

$$F_Z(z) \equiv \Phi(z)$$

(a) $P[-1 \le Z \le 1]$ $\underset{a}{\quad} \underset{b}{\quad}$

$= F_Z(b) - F_Z(a) = F_Z(1) - F_Z(-1) \equiv \Phi(1) - \Phi(-1)$

$= 0.8413 - (1 - \Phi(1)) = 0.8413 - (1 - 0.8413) = 0.6826$

(b) $P[-2 \le Z \le 2] = F_Z(2) - F_Z(-2) = \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2))$

$= 2\Phi(2) - 1 = 2(0.97725) - 1 = 0.9545$

Example 10.62. Suppose $X \sim \mathcal{N}(1, 2)$ ← σ^2. Find $P[1 \le X \le 2]$.

$$P[\underset{a}{1} \le x \le \underset{b}{2}] = F_X(b) - F_X(a) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)$$

$$= \Phi\left(\frac{2-1}{\sqrt{2}}\right) - \Phi\left(\frac{1-1}{\sqrt{2}}\right) = \Phi\left(\frac{1}{\sqrt{2}}\right) - \Phi(0)$$

$$= \Phi(0.71) - \Phi(0) \approx 0.7611 - 0.5 = 0.2611$$

Example 10.63. Signal Detection: Assume that in the detection of a digital signal, the background noise follows a normal distri- $N \sim \mathcal{N}(0, 0.45^2)$ bution with a mean of 0 volt and standard deviation of 0.45 volt. The system assumes a digital 1 has been transmitted when the

$$\Phi(6) \approx 1 - 0 \approx 1 \qquad \Phi(-6) \approx 0$$

$$Q(6) \to 0 \qquad \underset{cdf}{\quad} \qquad Q(-6) \approx 1$$

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z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 4: The standard normal CDF: Φ(z)

$P[\text{detecting a "1"}] = P[N > 0.9] = 1 - \widetilde{P[N \leq 0.9]} = 1 - F_N(0.9)$

(assuming nothing was sent)

$= 1 - \Phi\left(\frac{0.9-0}{0.45}\right) = 1 - \Phi(2) \approx 1 - 0.97725$

$= 0.02275$

voltage exceeds 0.9. (Otherwise, it assumes a digital 0 has been transmitted)

What is the probability of detecting a digital 1 when none was sent? [Montgomery and Runger, 2013, Ex. 4-15]

10.64. **Q-function**: $Q(z) = \int\limits_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ corresponds to $P[Z > z]$ where $Z \sim \mathcal{N}(0,1)$; that is $Q(z)$ is the probability of the "tail" of $\mathcal{N}(0,1)$. The Q function is then a complementary cdf (ccdf).

$f_Z(z)$

$P[Z \leq z] = \Phi(z)$

$Q(z) = P[Z > z]$

Figure 32: Q-function
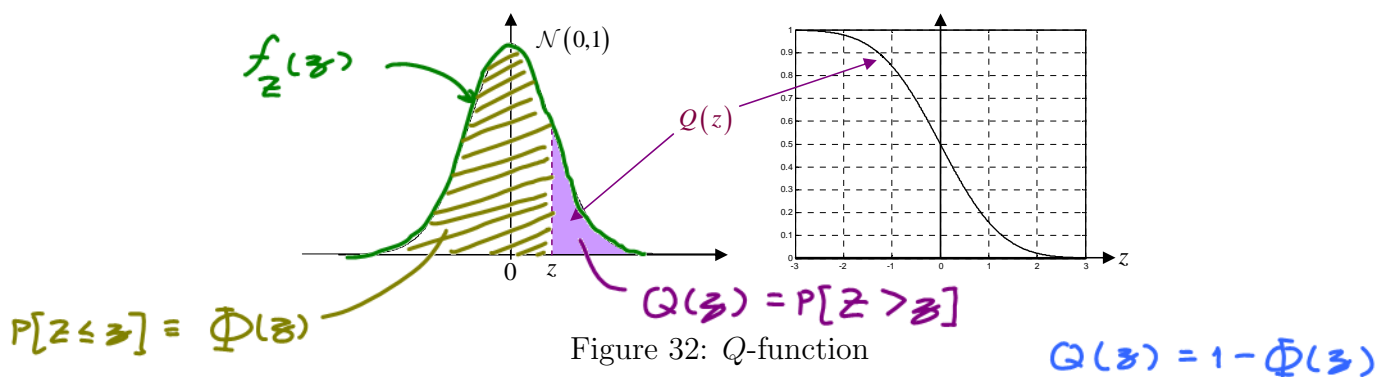
$Q(z) = 1 - \Phi(z)$

(a) Q is a decreasing function with $Q(0) = \frac{1}{2}$.

(b) $Q(-z) = 1 - Q(z) = \Phi(z)$

(c) Table 5 lists the values of $Q(z)$ for z between 3 to 5.

- For z between 0 to 3, we use $Q(z) = 1 - \Phi(z)$.
- For $z \geq 5$, the value of $Q(z)$ is extremely small. We may assume $Q(z) \approx 0$.

10.65. **Error function** (MATLAB): $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int\limits_{0}^{z} e^{-x^2} dx = 1 - 2Q\left(\sqrt{2}z\right)$

(a) It is an odd function of z.

(b) For $z \geq 0$, it corresponds to $P[|X| < z]$ where $X \sim \mathcal{N}\left(0, \frac{1}{2}\right)$.

(c) $\lim\limits_{z \to \infty} \mathrm{erf}(z) = 1$

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z	Q(z)	z	Q(z)	z	Q(z)	z	Q(z)	z	Q(z)
3.00	1.35E-03	3.40	3.37E-04	3.80	7.23E-05	4.20	1.33E-05	4.60	2.11E-06
3.01	1.31E-03	3.41	3.25E-04	3.81	6.95E-05	4.21	1.28E-05	4.61	2.01E-06
3.02	1.26E-03	3.42	3.13E-04	3.82	6.67E-05	4.22	1.22E-05	4.62	1.92E-06
3.03	1.22E-03	3.43	3.02E-04	3.83	6.41E-05	4.23	1.17E-05	4.63	1.83E-06
3.04	1.18E-03	3.44	2.91E-04	3.84	6.15E-05	4.24	1.12E-05	4.64	1.74E-06
3.05	1.14E-03	3.45	2.80E-04	3.85	5.91E-05	4.25	1.07E-05	4.65	1.66E-06
3.06	1.11E-03	3.46	2.70E-04	3.86	5.67E-05	4.26	1.02E-05	4.66	1.58E-06
3.07	1.07E-03	3.47	2.60E-04	3.87	5.44E-05	4.27	9.77E-06	4.67	1.51E-06
3.08	1.04E-03	3.48	2.51E-04	3.88	5.22E-05	4.28	9.34E-06	4.68	1.43E-06
3.09	1.00E-03	3.49	2.42E-04	3.89	5.01E-05	4.29	8.93E-06	4.69	1.37E-06
3.10	9.68E-04	3.50	2.33E-04	3.90	4.81E-05	4.30	8.54E-06	4.70	1.30E-06
3.11	9.35E-04	3.51	2.24E-04	3.91	4.61E-05	4.31	8.16E-06	4.71	1.24E-06
3.12	9.04E-04	3.52	2.16E-04	3.92	4.43E-05	4.32	7.80E-06	4.72	1.18E-06
3.13	8.74E-04	3.53	2.08E-04	3.93	4.25E-05	4.33	7.46E-06	4.73	1.12E-06
3.14	8.45E-04	3.54	2.00E-04	3.94	4.07E-05	4.34	7.12E-06	4.74	1.07E-06
3.15	8.16E-04	3.55	1.93E-04	3.95	3.91E-05	4.35	6.81E-06	4.75	1.02E-06
3.16	7.89E-04	3.56	1.85E-04	3.96	3.75E-05	4.36	6.50E-06	4.76	9.68E-07
3.17	7.62E-04	3.57	1.78E-04	3.97	3.59E-05	4.37	6.21E-06	4.77	9.21E-07
3.18	7.36E-04	3.58	1.72E-04	3.98	3.45E-05	4.38	5.93E-06	4.78	8.76E-07
3.19	7.11E-04	3.59	1.65E-04	3.99	3.30E-05	4.39	5.67E-06	4.79	8.34E-07
3.20	6.87E-04	3.60	1.59E-04	4.00	3.17E-05	4.40	5.41E-06	4.80	7.93E-07
3.21	6.64E-04	3.61	1.53E-04	4.01	3.04E-05	4.41	5.17E-06	4.81	7.55E-07
3.22	6.41E-04	3.62	1.47E-04	4.02	2.91E-05	4.42	4.94E-06	4.82	7.18E-07
3.23	6.19E-04	3.63	1.42E-04	4.03	2.79E-05	4.43	4.71E-06	4.83	6.83E-07
3.24	5.98E-04	3.64	1.36E-04	4.04	2.67E-05	4.44	4.50E-06	4.84	6.49E-07
3.25	5.77E-04	3.65	1.31E-04	4.05	2.56E-05	4.45	4.29E-06	4.85	6.17E-07
3.26	5.57E-04	3.66	1.26E-04	4.06	2.45E-05	4.46	4.10E-06	4.86	5.87E-07
3.27	5.38E-04	3.67	1.21E-04	4.07	2.35E-05	4.47	3.91E-06	4.87	5.58E-07
3.28	5.19E-04	3.68	1.17E-04	4.08	2.25E-05	4.48	3.73E-06	4.88	5.30E-07
3.29	5.01E-04	3.69	1.12E-04	4.09	2.16E-05	4.49	3.56E-06	4.89	5.04E-07
3.30	4.83E-04	3.70	1.08E-04	4.10	2.07E-05	4.50	3.40E-06	4.90	4.79E-07
3.31	4.66E-04	3.71	1.04E-04	4.11	1.98E-05	4.51	3.24E-06	4.91	4.55E-07
3.32	4.50E-04	3.72	9.96E-05	4.12	1.89E-05	4.52	3.09E-06	4.92	4.33E-07
3.33	4.34E-04	3.73	9.57E-05	4.13	1.81E-05	4.53	2.95E-06	4.93	4.11E-07
3.34	4.19E-04	3.74	9.20E-05	4.14	1.74E-05	4.54	2.81E-06	4.94	3.91E-07
3.35	4.04E-04	3.75	8.84E-05	4.15	1.66E-05	4.55	2.68E-06	4.95	3.71E-07
3.36	3.90E-04	3.76	8.50E-05	4.16	1.59E-05	4.56	2.56E-06	4.96	3.52E-07
3.37	3.76E-04	3.77	8.16E-05	4.17	1.52E-05	4.57	2.44E-06	4.97	3.35E-07
3.38	3.62E-04	3.78	7.84E-05	4.18	1.46E-05	4.58	2.32E-06	4.98	3.18E-07
3.39	3.49E-04	3.79	7.53E-05	4.19	1.39E-05	4.59	2.22E-06	4.99	3.02E-07

Table 5: The standard normal complementary CDF: $Q(z)$

(d) $\mathrm{erf}\,(-z) = -\mathrm{erf}\,(z)$

(e) $\Phi(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{(2)}}\right)\right) = \frac{1}{2}\mathrm{erfc}\left(-\frac{x}{\sqrt{2}}\right)$

(f) The complementary error function:
$\mathrm{erfc}\,(z) = 1 - \mathrm{erf}\,(z) = 2Q\left(\sqrt{2}z\right) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}\,dx$
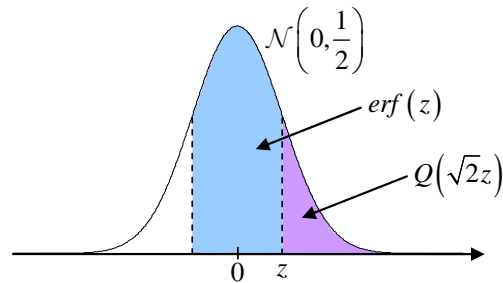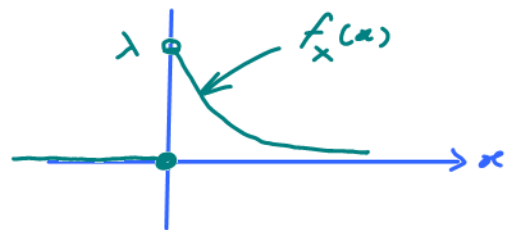

Figure 33: erf-function and Q-function

10.4.3 Exponential Distribution

Definition 10.66. The exponential distribution is denoted by $\mathcal{E}(\lambda)$.

(a) $\lambda > 0$ is a parameter of the distribution, often called the **rate parameter**.

(b) Characterized by

- $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$

- $F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$

$F_x(x) \equiv P[X \le x] = \int_{-\infty}^{x} f_x(t)\,dt = \int_0^x \lambda e^{-\lambda t}\,dt = \lambda \frac{e^{-\lambda t}}{(-\lambda)}\Big|_0^x$

$\quad x > 0$

$= -e^{-\lambda x} - (-1) = 1 - e^{-\lambda x}$

$\left(\begin{array}{l} \mathrm{ccdf} = 1 - \mathrm{cdf} = P[X > x] = 163e^{-\lambda x} \\ \uparrow \\ \text{complementary} \end{array}\right.$

(c) `MATLAB`:

- $X = $ `exprnd(1/`λ`)` or `random('exp',1/`λ`)`
- $f_X(x) = $ `exppdf(x,1/`λ`)` or `pdf('exp',x,1/`λ`)`
- $F_X(x) = $ `expcdf(x,1/`λ`)` or `cdf('exp',x,1/`λ`)`

10.67. The exponential distribution is intimately related to the Poisson process. In fact, the random variable X that equals the "distance" (or length or duration) between (any) successive events of a Poisson process with parameter λ is an exponential random variable with the same parameter.

Example 10.68. Exponential distribution is often used as a probability model for the (waiting) time until the next "rare" event occurs.

- time elapsed until the next earthquake in a certain region

- decay time of a radioactive particle

- time between independent events such as arrivals at a service facility or arrivals of customers in a shop.

- duration of a cell-phone call

- time it takes a computer network to transmit a message from one node to another.

10.69. In Example 10.37, we showed that $\mathbb{E}X = \frac{1}{\lambda}$.

Example 10.70. Suppose $X \sim \mathcal{E}(\lambda)$, find $P[1 < X < 2]$.

Method 1: $P[1<X<2] = \int_{1}^{2} f_X(x)\,dx$

Method 2: $P[1<X<2] = F_X(2) - F_X(1) = \left(1 - e^{-\lambda 2}\right) - \left(1 - e^{-\lambda 1}\right)$
$$= e^{-\lambda} - e^{-2\lambda}$$

10.71. Survival-, survivor-, or reliability-function:

Exercise 10.72. Exponential random variable as a continuous version of geometric random variable: Suppose $X \sim \mathcal{E}(\lambda)$. Show that $\lfloor X \rfloor \sim \mathcal{G}_0(e^{-\lambda})$ and $\lceil X \rceil \sim \mathcal{G}_1(e^{-\lambda})$

Example 10.73. Phone Company A charges \$0.15 per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone Company B also charges \$0.15 per minute. However, Phone Company B calculates its charge based on the exact duration of a call. If T, the duration of a call in minutes, is exponential with parameter $\lambda = 1/3$, what are the expected revenues per call $\mathbb{E}[R_A]$ and $\mathbb{E}[R_B]$ for companies A and B?

 Solution: First, note that $\mathbb{E}T = \frac{1}{\lambda} = 3$. Hence,

$$\mathbb{E}[R_B] = \mathbb{E}[0.15 \times T] = 0.15\mathbb{E}T = \$0.45.$$

and

$$\mathbb{E}[R_A] = \mathbb{E}[0.15 \times \lceil T \rceil] = 0.15\mathbb{E}\lceil T \rceil.$$

Now, recall, from Exercise 10.72, that $\lceil T \rceil \sim \mathcal{G}_1(e^{-\lambda})$. Hence, $\mathbb{E}\lceil T \rceil = \frac{1}{1-e^{-\lambda}} \approx 3.53$. Therefore,

$$\mathbb{E}[R_A] = 0.15\mathbb{E}\lceil T \rceil \approx 0.5292.$$

10.74. *Memoryless property*: The exponential r.v. is the only continuous[48] r.v. on $[0, \infty)$ that satisfies the memoryless property:

$$P[X > s + x \mid X > s] = P[X > x]$$

for all $x > 0$ and all $s > 0$ [18, p. 157–159]. In words, the future is independent of the past. The fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen.

- Imagining that the exponentially distributed random variable X represents the lifetime of an item, the residual life of an item has the same exponential distribution as the original lifetime, regardless of how long the item has been already in use. In other words, there is no deterioration/degradation over time. If it is still currently working after 20 years of use, then today, its condition is "just like new".

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[48]For discrete random variable, geometric random variables satisfy the memoryless property.

- In particular, suppose we define the set $B+x$ to be $\{x + b : b \in B\}$. For any $x > 0$ and set $B \subset [0, \infty)$, we have

$$P[X \in B + x | X > x] = P[X \in B]$$

because

$$\frac{P[X \in B + x]}{P[X > x]} = \frac{\int_{B+x} \lambda e^{-\lambda t} dt}{e^{-\lambda x}} \overset{\tau = t - x}{=\!=\!=} \frac{\int_B \lambda e^{-\lambda(\tau + x)} d\tau}{e^{-\lambda x}}.$$

Example 10.75. The exponential distribution is often used in reliability studies as the model for the time until failure of a device. For example, the lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours.

The lack of memory property of the exponential distribution implies that the device does not wear out. That is, regardless of how long the device has been operating, the probability of a failure in the next 1000 hours is the same as the probability of a failure in the first 1000 hours of operation.

10.76. The lifetime L of a device with failures caused by random shocks might be appropriately modeled as an exponential random variable. However, the lifetime L of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by other distributions such as the Weibull distribution.

10.77. Summary:

$X \sim$	Support S_X	$f_X(x) =$
Uniform $\mathcal{U}(a,b)$	(a,b)	$\begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$
Normal (Gaussian) $\mathcal{N}(m, \sigma^2)$	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$
Exponential $\mathcal{E}(\lambda)$	$(0, \infty)$	$\begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$

Table 6: Examples of probability density functions. Here, $\lambda, \sigma > 0$.

Last time, we study an example of a family of
continuous RV.

$$\text{Gaussian RV} \rightarrow X \sim \mathcal{N}(m, \sigma^2) \leftarrow \text{Gaussian distribution}$$

mean (expected value)   variance

pdf of a Gaussian RV

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

We know that, for any $f_X(x)$,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}\,dx = 1$$

cdf

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(t)\,dt \leftarrow \text{most of the time, this can not be simplified algebraically}$$

First, we looked at a special case

standard Gaussian RV

$$\rightarrow Z \sim \mathcal{N}(0, 1)$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}$$

$$\Phi(z) \equiv F_Z(z) = \int_{-\infty}^{z} f_Z(t)\,dt$$

$$P[a < Z < b] = F_Z(b) - F_Z(a) = \Phi(b) - \Phi(a)$$

Next, we returned to the general case

$$X \sim \mathcal{N}(m, \sigma^2)$$

$$F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$$

$$P[a < X < b] = F_X(b) - F_X(a) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)$$

	Discrete	Continuous
Definition	$\equiv$ Can find a countable support $\equiv F_X(x)$ is a staircase function	$\equiv P[X = x] = 0$ for all x $\equiv F_X(x)$ is a cont. function
(Default) Support	$\{x : p_X(x) > 0\}$	$\{x : f_X(x) > 0\}$
$P^X(B) \equiv P[X \in B] =$	$\sum_{x \in B} p_X(x)$	$\int_B f_X(x) dx$
$P[X = x] =$	$p_X(x) = F_X(x) - F_X(x^-)$	0
Interval prob.	$P^X((a,b]) = F_X(b) - F_X(a)$ $P^X([a,b]) = F_X(b) - F_X(a^-)$ $P^X([a,b)) = F_X(b^-) - F_X(a^-)$ $P^X((a,b)) = F_X(b^-) - F_X(a)$	$P^X((a,b]) = P^X([a,b])$ $= P^X([a,b)) = P^X((a,b))$ $= \int_a^b f_X(x)dx = F(b) - F(a)$
$\mathbb{E}X =$	$\sum_x x p_X(x)$	$\int_{-\infty}^{+\infty} x f_X(x) dx$
For $Y = g(X)$,	$p_Y(y) = \sum_{x : g(x) = y} p_X(x)$	$f_Y(y) = \dfrac{d}{dy} P[g(X) \le y].$ Alternatively, $f_Y(y) = \sum_k \dfrac{f_X(x_k)}{\|g'(x_k)\|},$ x_k are the real-valued roots of the equation $y = g(x)$.
For $Y = g(X)$, $P[Y \in B] =$	$\sum_{x : g(x) \in B} p_X(x)$	$\int_{\{x : g(x) \in B\}} f_X(x) dx$
$\mathbb{E}[g(X)] =$	$\sum_x g(x) p_X(x)$	$\int_{-\infty}^{+\infty} g(x) f_X(x) dx$
$\mathbb{E}[X^2] =$	$\sum_x x^2 p_X(x)$	$\int_{-\infty}^{+\infty} x^2 f_X(x) dx$
$\text{Var} X =$	$\sum_x (x - \mathbb{E}X)^2 p_X(x)$	$\int_{-\infty}^{+\infty} (x - \mathbb{E}X)^2 f_X(x) dx$

Table 7: Important Formulas for Discrete and Continuous Random Variables